# Workshop

## Multivariate General Linear Models
## (# 170, Richard F. Haase)

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university of groningen

# Overview

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# Goals of today's talk

Four main goals motivated the organization of this talk:

1. *Provide insight into the generalization of univariate multiple regression (UMR) to multivariate multiple regression (MMR).*

   We will see that the model formulation, parameter estimation, and inferential procedures of MMR extend naturally from UMR.

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2. *Look into the most popular multivariate test statistics in use (Pillai's trace $V$, Wilks's $\Lambda$, Hotteling's trace $T$, and Roy's greatest characteristic root $\theta$).*

   We will relate each of these multivariate test statistics to common tools and concepts from UMR (namely, $F$ tests, *sr*'s, and *pr*'s).

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3. *Present a multivariate version of $R^2$ in MMR, here called $R_m^2$.*

   This scalar $R^2$ measure is not commonly used in MMR, but it is useful as a summarized measure of "explained variance" across the set of all DVs.

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4. *(Briefly) Refer to MANOVA as a special case of MMR.*

# Overview

# Review of univariate GLMs

Multiple regression model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_q X_q + \varepsilon$$

$X_1, \ldots, X_q$: Predictors (continuous and/or categorical).

$(\beta_0)$, $\beta_1, \ldots, \beta_q$: Regression coefficients.

$Y$: Dependent variable (only one in univariate MR).

$\varepsilon$: Error term.

Expressing the same model in matrix algebraic terms:

$$\mathbf{y}_{n\times 1} = \mathbf{X}_{n\times(q+1)}\ \boldsymbol{\beta}_{(q+1)\times 1} + \boldsymbol{\varepsilon}_{n\times 1}$$

$n$: Sample size.

$q$: Number of predictors.

# Review of univariate GLMs

$$\mathbf{y}_{n\times 1} = \mathbf{X}_{n\times(q+1)}\ \boldsymbol{\beta}_{(q+1)\times 1} + \boldsymbol{\varepsilon}_{n\times 1}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}_{n\times 1} = \underbrace{\begin{bmatrix} 1 & X_{11} & X_{12} & \cdots & X_{1q} \\ 1 & X_{21} & X_{22} & \cdots & X_{2q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{n1} & X_{n2} & \cdots & X_{nq} \end{bmatrix}_{n\times(q+1)}}_{\text{Design matrix}} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_q \end{bmatrix}_{(q+1)\times 1} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}_{n\times 1}$$

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Four steps of GLM analysis:

1. Specify the model.
2. Estimate the model parameters.
3. Check goodness of fit of the model.
4. Test hypotheses about the model.

# Review of univariate GLMs

1. *Specify the model.*
   - Choose adequate predictors $X_i$ $(i = 1, 2, \ldots, q)$ and DV $Y$ (choice is typically theory-driven, not statistics-driven).
   - Fill in $\mathbf{y}_{n\times 1}$ and the design matrix $\mathbf{X}_{n\times(q+1)}$ correctly. Coding categorical predictors required.

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2. *Estimate the model parameters.*
   The OLS solution consists of finding $\boldsymbol{\beta}$ that minimizes the sum of the squared residuals:

$$\sum_{i=1}^{n} \varepsilon_i^2 = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) = \boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}.$$

   The solution is:
   - Unstandardized regression coefficients:

$$\widehat{\boldsymbol{\beta}}_{(q+1)\times 1} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}.$$

   - Standardized regression coefficients ('beta' coefficients):

$$\widehat{\boldsymbol{\beta}}^*_{(q+1)\times 1} = (\mathbf{Z}_\mathbf{X}'\mathbf{Z}_\mathbf{X})^{-1}\mathbf{Z}_\mathbf{X}'\mathbf{Z}_\mathbf{y} = \mathbf{R}_{XX}^{-1}\mathbf{R}_{XY}.$$

# Review of univariate GLMs

3. *Check goodness of fit of the model.*

$$SS_{\text{Total}} = SS_{\text{Model}} \qquad + SS_{\text{Error}}$$
$$\sum_{i=1}^{n} \left(Y_i - \overline{Y}\right)^2 = \sum_{i=1}^{n} \left(\widehat{Y}_i - \overline{Y}\right)^2 + \sum_{i=1}^{n} \left(Y_i - \widehat{Y}_i\right)^2$$

The most common measure of goodness of fit is

$$\begin{aligned} R^2 &= \frac{SS_{\text{Model}}}{SS_{\text{Total}}} \\ &= 1 - \frac{SS_{\text{Error}}}{SS_{\text{Total}}} \\ &= \widehat{\beta}_1^* r_{Y \cdot X1} + \widehat{\beta}_2^* r_{Y \cdot X2} + \cdots + \widehat{\beta}_q^* r_{Y \cdot Xq}. \end{aligned}$$

# Review of univariate GLMs

Other goodness of fit measures include the semipartial (*sr*) and partial (*pr*) correlation coefficients.

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- Semipartial correlation = Correlation between Y and the part of the predictor that is uncorrelated to all the remaining predictors.
  Equivalently, it is the $R^2$ increment achieved by adding the predictor to a model that already includes the remaining $(q-1)$ predictors.
  E.g.,

$$\begin{aligned} sr_1^2 &= r^2(Y, X_1 | X_2 \cdots X_q). \\ &= R^2_{Y \cdot X_1 \cdots X_q} - R^2_{Y \cdot X_2 \cdots X_q} \\ &= R^2_{\text{full}} - R^2_{\text{restricted}}. \end{aligned}$$

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- Partial correlation = Correlation between the Y and a predictor, after the remaining predictors have been partialled out from both. E.g.,

$$pr_1 = r(Y | X_2 \cdots X_q, X_1 | X_2 \cdots X_q).$$

# Review of univariate GLMs

4. *Test hypotheses about the model.*
   Under the usual regression assumptions (namely of normality) then:

- $F$ test associated to $R^2$:

$$F = \frac{SS_{\text{Model}}/q}{SS_{\text{Error}}/(n-q-1)} = \frac{MS_{\text{Model}}}{MS_{\text{Error}}} \underset{H_0}{\sim} F(q, n-q-1).$$

- More generally, the $F$ test associated to $sr^2$ (or $\beta$):

$$F = \frac{(R^2_{\text{full}} - R^2_{\text{restricted}})/(q-q_r)}{(1-R^2_{\text{full}})/(n-q-1)} \underset{H_0}{\sim} F(q-q_r, n-q-1),$$

where $q_r$ = number of predictors in the restricted model.

**Obs:** Any test of interest in regression (i.e., any contrast) can be reexpressed as an $F$ test as shown above. So there is only one type of test, really.

# Review of univariate GLMs

In general, any contrast of interest under the GLM can be written as follows:

$$H_0 : \mathbf{L}_{c\times(q+1)}\boldsymbol{\beta}_{(q+1)\times 1} = \mathbf{0}_{c\times 1},$$

where $c =$ number of contrasts to be tested and $\mathbf{L}$ is a matrix of contrast coefficients.

Example:

- $H_0 : \beta_1 = \cdots = \beta_q = 0$ becomes $(c = q)$

$$H_0 : \mathbf{L}\boldsymbol{\beta} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_q \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

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# Review of univariate GLMs

Example:

- $H_0 : \beta_1 = 0$ becomes $(c = 1)$

$$H_0 : \mathbf{L}\boldsymbol{\beta} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_q \end{bmatrix} = \beta_1 = 0$$

- $H_0 : \beta_1 = \beta_2$ becomes $(c = 1)$

$$H_0 : \mathbf{L}\boldsymbol{\beta} = \begin{bmatrix} 0 & 1 & -1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_q \end{bmatrix} = \beta_1 - \beta_2 = 0$$

# Review of univariate GLMs

**In general:**

Any hypothesis of the type

$$H_0 : \mathbf{L}_{c\times(q+1)}\boldsymbol{\beta}_{(q+1)\times 1} = \mathbf{0}_{c\times 1}$$

can be tested by means of the $F$ test

$$F = \frac{(R^2_{\text{full}} - R^2_{\text{restricted}})/c}{(1-R^2_{\text{full}})/(n-q-1)} = \frac{SS_{\text{Hypothesis}}/c}{SS_{\text{Error}}/(n-q-1)} \underset{H_0}{\sim} F(c, n-q-1),$$

with $R^2_{\text{full}}$ and $R^2_{\text{restricted}}$ computed directly from $\mathbf{X}$, $\widehat{\boldsymbol{\beta}}$, and suitable $\mathbf{L}$ matrices.

. . .

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# Overview

# Structure of multivariate GLMs

Extend the previous results to cases with more than one DV.

$$\mathbf{Y}_{n\times p} = \mathbf{X}_{n\times(q+1)}\ \mathbf{B}_{(q+1)\times p} + \mathbf{E}_{n\times p}$$

*n*: Sample size.

*q*: Number of predictors.

*p*: Number of DVs.

The design matrix $\mathbf{X}$ is the same as before.
$\mathbf{Y}$, $\mathbf{B}$, and $\mathbf{E}$ were extended to accomodate $p$ columns.

# Overview

# Estimating the parameters of the multivariate GLM

$$\mathbf{Y}_{n \times p} = \mathbf{X}_{n \times (q+1)} \; \mathbf{B}_{(q+1) \times p} + \mathbf{E}_{n \times p}$$

The OLS solution consists of finding **B** that minimizes the sum of the squared residuals:

$$tr(\mathbf{E}'\mathbf{E}) = tr\left[(\mathbf{Y} - \mathbf{XB})'(\mathbf{Y} - \mathbf{XB})\right].$$

The solution is:

- Unstandardized regression coefficients:

$$\widehat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}.$$

- Standardized regression coefficients ('beta' coefficients):

$$\widehat{\mathbf{B}}^* = (\mathbf{Z}_{\mathbf{X}}'\mathbf{Z}_{\mathbf{X}})^{-1}\mathbf{Z}_{\mathbf{X}}'\mathbf{Z}_{\mathbf{Y}} = \mathbf{R}_{XX}^{-1}\mathbf{R}_{XY}.$$

# Overview

# Partitioning the SSCP, strength of assoc., test statistics

Recall that for univariate GLMs,

$$\widehat{\mathbf{y}}_{n\times 1} = \mathbf{X}_{n\times(q+1)}\ \widehat{\boldsymbol{\beta}}_{(q+1)\times 1}$$

with $\widehat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$.

The SS partitioning is given by

$$\begin{aligned} SS_{\text{Total}} &= SS_{\text{Model}} && + SS_{\text{Error}} \\ \sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2 &= \sum_{i=1}^{n}\left(\widehat{Y}_i - \overline{Y}\right)^2 && + \sum_{i=1}^{n}\left(Y_i - \widehat{Y}_i\right)^2 \\ (\mathbf{y}'\mathbf{y} - n\overline{\mathbf{y}}'\overline{\mathbf{y}}) &= (\widehat{\mathbf{y}}'\mathbf{y} - n\overline{\mathbf{y}}'\overline{\mathbf{y}}) && + (\mathbf{y}'\mathbf{y} - \widehat{\mathbf{y}}'\mathbf{y}) \end{aligned}$$

Multivariate GLMs generalize these formulas to accomodate multiple DVs (say, $p$).

# Partitioning the SSCP, strength of assoc., test statistics

For multivariate GLMs,

$$\widehat{\mathbf{Y}}_{n\times p} = \mathbf{X}_{n\times(q+1)}\ \widehat{\mathbf{B}}_{(q+1)\times p}$$

with $\widehat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$.

The SSCP (matrices!) partitioning is given by

$$\begin{aligned} SSCP_{\text{Total}} &= SSCP_{\text{Model}} + SSCP_{\text{Error}} \\ \underbrace{\left(\mathbf{Y}'\mathbf{Y} - n\overline{\mathbf{Y}}'\overline{\mathbf{Y}}\right)}_{p\times p} &= \underbrace{\left(\widehat{\mathbf{Y}}'\mathbf{Y} - n\overline{\mathbf{Y}}'\overline{\mathbf{Y}}\right)}_{p\times p} + \underbrace{\left(\mathbf{Y}'\mathbf{Y} - \widehat{\mathbf{Y}}'\mathbf{Y}\right)}_{p\times p} \end{aligned}$$

# Partitioning the SSCP, strength of assoc., test statistics

Recall that the $R^2$ measure of strength of association was, for univariate GLMs, given by

$$R^2 = \frac{SS_{\text{Model}}}{SS_{\text{Total}}} = 1 - \frac{SS_{\text{Error}}}{SS_{\text{Total}}}.$$

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The $R^2$ of each DV in multivariate GLMs is readily available using the same formula:

$$\left(R^2_{Y_1}, R^2_{Y_2}, \ldots, R^2_{Y_p}\right) = \frac{\text{Diag}(SSCP_{\text{Model}})}{\text{Diag}(SSCP_{\text{Total}})} = 1 - \frac{\text{Diag}(SSCP_{\text{Error}})}{\text{Diag}(SSCP_{\text{Total}})}$$

This is equivalent to running $p$ separate univariate GLMs.

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But we would like an overall, multivariate, measure of the strength of association between **Y** and **X** across the $p$ DVs. Only one value, not $p$ separate $R^2$ values...

Q: Is there such a measure?

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Q: Is there such a measure?

A: Well, yes... The problem is that there are several.

# Partitioning the SSCP, strength of assoc., test statistics

One first attempt for a multivariate measure of strength of association:

$$R^2_{dYX} = \frac{R^2_{Y_1} + R^2_{Y_2} + \cdots + R^2_{Y_p}}{p}.$$

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- Another problem:
  The redundancy index is asymmetric: $R^2_{dYX} \neq R^2_{dXY}$ unless $p = q$.
  This is awkward.
  (Think of overlapping areas in Venn diagrams.)

# Partitioning the SSCP, strength of assoc., test statistics

We want better measures of strength of association between **Y** and **X**, in particular:

- Adjusted for multicollinearity among the $Y_i$ variables.
- Symmetric.

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We want better measures of strength of association between **Y** and **X**, in particular:

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**Motivation:** Generalize the univariate idea of a $sr^2$,

$$sr^2 = R^2_{\text{full}} - R^2_{\text{restricted}},$$

and its associated $F$ test:

$$F = \frac{SS_{\text{Hypothesis}}/(q - q_r)}{SS_{\text{Error}}/(n - q - 1)} \underset{H_0}{\sim} F(q - q_r, n - q - 1).$$

Recall univariate

# Partitioning the SSCP, strength of assoc., test statistics

For full model (i.e., based on all $q$ predictors $X_1, \ldots, X_q$):

$$\mathbf{Y} = \mathbf{XB} \qquad + \mathbf{E}$$
$$\underbrace{SSCP_{\text{Total}}}_{\mathbf{Q}_T} = \underbrace{SSCP_{\text{Full}}}_{\mathbf{Q}_F} + \underbrace{SSCP_{\text{Error}}}_{\mathbf{Q}_E}$$

For reduced model (i.e., based on only a subset of predictors):

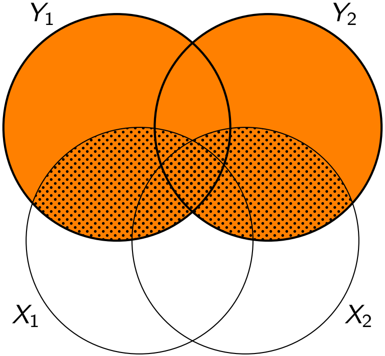
$$\mathbf{Y} = \mathbf{X}_R\mathbf{B}_R \qquad + \mathbf{E}'$$
$$\underbrace{SSCP_{\text{Total}}}_{\mathbf{Q}_T} = \underbrace{SSCP_{\text{Restricted}}}_{\mathbf{Q}_R} + \underbrace{SSCP_{\text{Error}}}_{\mathbf{Q}_{E'}}$$

Thus, focus on the hypothesis SSCP matrix:

$$\mathbf{Q}_H = \mathbf{Q}_F - \mathbf{Q}_R.$$

$\mathbf{Q}_H$ = Incremental influence of the variables in the full model that are not in the restricted model.

# Partitioning the SSCP, strength of assoc., test statistics




$$\text{“}R^2\text{”} = \frac{\mathbf{Q}_F}{\mathbf{Q}_T} = \frac{\mathbf{Q}_F}{\mathbf{Q}_F + \mathbf{Q}_E}$$

. . .

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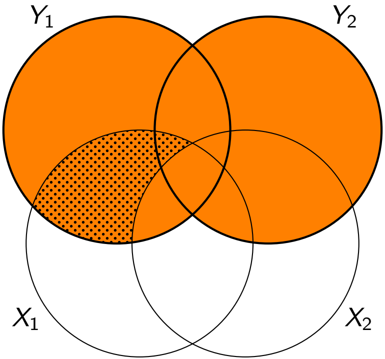
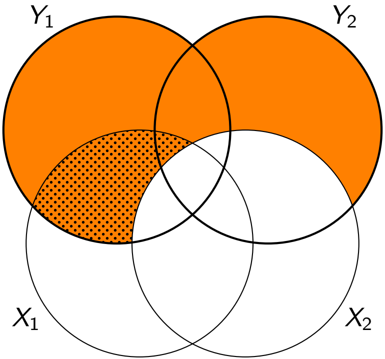



$$\text{“}sr_1^2\text{”} = \frac{\mathbf{Q}_H}{\mathbf{Q}_T} = \frac{\mathbf{Q}_H}{\mathbf{Q}_F + \mathbf{Q}_E},$$

where $\mathbf{Q}_H$ represents the unique contribution of $X_1$ to explaining the total variance in $\mathbf{Y}$.

# Partitioning the SSCP, strength of assoc., test statistics




$$\text{“}pr_1^2\text{”} = \frac{\mathbf{Q}_H}{\mathbf{Q}_H + \mathbf{Q}_E},$$

where $\mathbf{Q}_H$ represents the unique contribution of $X_1$ to explaining the variance in $\mathbf{Y}$ not explained by $X_2$.

. . .

# Partitioning the SSCP, strength of assoc., test statistics

A second attempt for a multivariate measure of strength of association: Hooper's squared trace correlation. From

$$\text{``}R^2\text{''} = \frac{\mathbf{Q}_F}{\mathbf{Q}_T}$$

Recall Venn's diagram

one derives

$$\text{``}R^2\text{''} = \mathbf{Q}_T^{-1}\mathbf{Q}_F,$$

which is still a $(p \times p)$ matrix.

# Partitioning the SSCP, strength of assoc., test statistics

A second attempt for a multivariate measure of strength of association: Hooper's squared trace correlation. From

$$``R^2" = \frac{\mathbf{Q}_F}{\mathbf{Q}_T}$$

Recall Venn's diagram

one derives

$$``R^2" = \mathbf{Q}_T^{-1}\mathbf{Q}_F,$$

which is still a $(p \times p)$ matrix.

Hooper (1959) suggested the following (scalar) formula:

$$\bar{r}^2 = \frac{1}{p}\text{tr}\left(\mathbf{Q}_T^{-1}\mathbf{Q}_F\right) = \frac{1}{p}\text{tr}\left(\mathbf{R}_{YY}^{-1}\mathbf{R}_{YX}\mathbf{R}_{XX}^{-1}\mathbf{R}_{XY}\right)$$

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We are finally led to present the "big four" $R^2$-like measures of wide use nowadays, all of which are symmetric and adjusted for multicollinearity among the $Y$s:

- Pillai's trace $V$.
- Wilks' $\Lambda$.
- Lawley-Hotteling's trace $T$.
- Roy's GCR $\theta$.

# Partitioning the SSCP, strength of assoc., test statistics

| Test statistic | Multivariate test statistic | $R_m^2$ | Univ. $R^2$ conceptual equivalent | Multiv. $F$-test equivalent |
|---|---|---|---|---|
| Pillai's $V$ | $V = \text{tr}\left[(\mathbf{Q}_H + \mathbf{Q}_E)^{-1}\,\mathbf{Q}_H\right]$ | $R_V^2 = \frac{V}{s}$ | $pr^2$ Venn diag. <br> $(=R^2$ if $\mathbf{Q}_F = \mathbf{Q}_H)$ | $(*)$ |
| Wilks' $\Lambda$ | $\Lambda = \frac{\lvert\mathbf{Q}_E\rvert}{\lvert\mathbf{Q}_H+\mathbf{Q}_E\rvert}$ | $R_\Lambda^2 = 1 - \Lambda^{\frac{1}{s}}$ | $1 - pr^2$ <br> $(=1 - R^2$ if $\mathbf{Q}_F = \mathbf{Q}_H)$ | $(*)$ |
| Hotelling's $T$ | $T = \text{tr}\left(\mathbf{Q}_E^{-1}\mathbf{Q}_H\right)$ | $R_T^2 = \frac{T}{T+s}$ | $\frac{pr^2}{1-pr^2}$ <br> $(=\frac{R^2}{1-R^2}$ if $\mathbf{Q}_F = \mathbf{Q}_H)$ | $(*)$ |
| Roy's $\theta$ | $\theta = \text{max}_{\text{eigen}}\left(\mathbf{Q}_E^{-1}\mathbf{Q}_H\right)$ | $R_\theta^2 = \frac{\theta}{1+\theta}$ | $r^2$ | $(*)$ |

$s = \min(p, q_h)$.
$\rho^2_{\text{max}}$ = maximum squared canonical correlation between $\mathbf{X}$ and $\mathbf{Y}$.

$(*)$ = These $F$ test are all similar to each other, and all are approximations of the exact tests based on $V$, $\Lambda$, $T$, and $\theta$.

. . .

# Partitioning the SSCP, strength of assoc., test statistics

**Example — Personality and success in job application process**
Based on Caldwell and Burger (1998).

| **Predictors** | |
|---|---|
| `Neurot` | Neuroticism |
| `Extrav` | Extraversion |
| `Consci` | Conscientiousness |
| **Outcomes** | |
| `BackPrep` | Background preparation for the interviews |
| `SociPrep` | Social preparation for the interviews |
| `FollowUp` | Number of follow-up interviews achieved |
| `Offers` | Number of offers of employment received |

- Original data based on an observational study of 99 college students.
- I generated synthetic data based on the original means, SDs, and correlations for the seven variables above.

# Partitioning the SSCP, strength of assoc., test statistics

**Multivariate tests**

| Test | Test Stat. | $R_m^2$ | Approx. $F$ | Num Df | Den Df | $p$-value |
|---|---|---|---|---|---|---|
| Pillai's $V$ | 0.470 | $\frac{V}{s} = .156$ | 4.361 | 12 | 282 | $<.001$ |
| Wilks' $\Lambda$ | 0.579 | $1 - \Lambda^{\frac{1}{s}} = .166$ | 4.658 | 12 | 243.701 | $<.001$ |
| Hotteling's T | 0.645 | $\frac{T}{T+s} = .177$ | 4.872 | 12 | 272 | $<.001$ |
| Roy's $\theta$ | 0.489 | $\frac{\theta}{1+\theta} = .328$ | 11.492 | 4 | 94 | $<.001$ |

In this example, $s = \min(p, q_h) = \min(4, 3) = 3$.

Obs.: $R_m^2$ values are typically not given by statistical software, so manual computation might be needed.

# Partitioning the SSCP, strength of assoc., test statistics

Some notes:

- The interpretation of $R_m^2$ for $V$, $\Lambda$, and $T$ is more or less the same, namely:

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- In cases where $R_\theta^2 \gg \{R_V^2, R_\Lambda^2, R_T^2\}$:
  Be careful not to put too much trust on $R_\theta^2$.

# Partitioning the SSCP, strength of assoc., test statistics

Both SPSS and R do not give the omnibus (approximate) $F$ test results. One needs to explicitly ask for these:

## In SPSS...

```
GLM BackPrep SociPrep FollowUp Offers WITH Neurot Extrav Consci
/LMATRIX Neurot 1; Extrav 1; Consci 1.
```

(see output table **Multivariate Test Results**)

## In R...

```
library(car)
# Below, the data frame is called 'Data.CB'.
res.CB       <- lm(cbind(BackPrep, SociPrep, FollowUp, Offers) ~
                      Neurot + Extrav + Consci, Data.CB)
L            <- cbind(0, diag(3))
TestStats.CB <- linearHypothesis(res.CB, L)
```

(see output table **Multivariate Tests**)

Recall omnibus contrast

# Overview

# Testing hypotheses in the multivariate GLM

All (approximate!) multivariate $F$-tests have the same form, which is an extension from the common univariate $F$ test for $pr^2$:

$$F = \frac{R^2_{\text{hypothesis}}/df_h}{(1 - R^2_{\text{hypothesis}})/df_e} \underset{H_0}{\sim} F(q - q_r, n - q - 1).$$

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The approximate multivariate $F$-tests ((*) in Recall multivariate tests ) are a test of the partial $R^2_m$:

$$F = \frac{R^2_m/v_h}{(1 - R^2_m)/v_e} \underset{H_0}{\sim} F(v_h, v_e),$$

with $v_h, v_e$ specific to each test statistic (Pillai, Wilks, etc.).

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  Exact test is preferred (but not often provided by software).

- It is straightforward to adapt these approximate multivariate $F$ tests to test any contrast of interest, similarly to what we saw for univariate models:

$$H_0 : \mathbf{L}_{c\times(q+1)}\mathbf{B}_{(q+1)\times p} = \mathbf{0}_{c\times p}$$

  with

$$\mathbf{Q}_H = (\mathbf{L}\widehat{\mathbf{B}})' \left(\mathbf{L}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{L}'\right)^{-1} (\mathbf{L}\widehat{\mathbf{B}}).$$

# Testing hypotheses in the multivariate GLM

For the running example (personality and success in job application process):

- $H_0$ : There is no overall effect of the three personality dimensions on the DVs (i.e., the omnibus test discussed before).

$$\mathbf{L} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In R:

```
L <- cbind(0, diag(3))
linearHypothesis(res.CB, L)
```

- $H_0$ : Extraversion (2nd predictor) has no effect.

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix},$$

so $H_0 : \mathbf{B}_{\text{Extr. on the 4 DVs}} = (0, 0, 0, 0)$.
In R:

```
L <- c(0, 0, 1, 0)
linearHypothesis(res.CB, L)
```

- ...

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- The $p$ univariate follow-up tests are based on DVs which are not adjusted for their mutual correlations. This may lead to univariate follow-up tests overestimating the contribution of single DVs to the multivariate relationship.
- The Roy-Bargman stepdown tests are one way to solve this issue related to correlated DVs.

# Overview

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  Reference (dummy) coding, unweighted effects coding, weighted effects coding, contrast coding, cell mean coding, ...
- Different coding systems lead to different regression coefficients $\mathbf{B}$.
- The multivariate test of contrasts (e.g., omnibus test, test for individual predictors, ...) is performed as before:

$$H_0 : \mathbf{L}_{c\times(q+1)}\mathbf{B}_{(q+1)\times p} = \mathbf{0}_{c\times p},$$

  where the specific form of $\mathbf{L}$ will depend on $\mathbf{B}$ (i.e., on the coding system of choice).

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- Scalar multivariate $R^2$s exist which can be computed and reported.
- MANOVA directly benefits from these insights.